

The odd twistor transform in eleven-dimensional supergravity

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Abstract

We define a twistor-like transform of the equations of eleven-dimensional supergravity. More precisely these equations are encoded by the CR-structure on the twistor space $\mathcal{P}^{2 \times 15 + 11 | 8 \times 2 + 16}$. In addition equations of the linearized eleven-dimensional supergravity adapted to the 3-form potential can be transformed into the tangential Cauchy-Riemann equation $\bar{\partial}\omega = 0$ on \mathcal{P} .

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1 Introduction

The classical superspace formulation ([9], [13]) makes the supersymmetries manifest, with a drawback that the fields it encodes are constrained to satisfy supergravity equations. The proposal of Cederwall [11] is supposed to rectify that. Cederwall's approach still depends on the choice of a background solution of supergravity equations, but the fields in his construction are unconstrained. In the flat background, the fields are elements of

$$\Theta = \Theta_{\mathbb{R}^{11|32}} = A \otimes \Lambda[\theta^1, \dots, \theta^{32}] \otimes C^\infty(\mathbb{R}^{11}). \quad (1)$$

The tensor factor A is the commutative algebra

$$\mathbb{R}[\lambda^1, \dots, \lambda^{32}]/(v^i) \quad (2)$$

with

$$v^i = \Gamma_{\alpha\beta}^i \lambda^\alpha \lambda^\beta, i = 1, \dots, 11, \alpha, \beta = 1, \dots, 32 \quad (3)$$

The algebra is graded by the degree in λ^α . We use summation convention over repeated indices: lower-case Greek letters run between 1 and 32, lower-case Roman letters have a range between 1 and 11, capital Greek letters run between 1 and 8. In addition to variables $\lambda^\alpha \in A$ and $\theta^\alpha \in \Lambda[\theta^1, \dots, \theta^{32}]$ it is convenient to fix coordinates x^1, \dots, x^{11} on \mathbb{R}^{11} . The variables $\lambda^\alpha, \theta^\alpha$ transform in a spinor representation $\mathfrak{s}_{10,1}^{\mathbb{R}}$ of the Lorentz group $\text{Spin}(10, 1, \mathbb{R})$. The coordinates x^i on $V^{10,1} \mathbb{R}$ transform as vectors under $\text{SO}(10, 1, \mathbb{R})$. The reader might wish to consult [14] for a mathematical introduction to spinors and Γ -matrices in the Lorentz signature.

The operator

$$D = \lambda^\alpha \eta_\alpha \quad (4)$$

where

$$\eta_\alpha = \frac{\partial}{\partial \theta^\alpha} - \Gamma_{\alpha\beta}^i \theta^\beta \frac{\partial}{\partial x^i} \quad (5)$$

is a differential in Θ . According to [28],[23],[4] the D -cohomology of Θ coincide with the space of solutions of the linearized equations of eleven-dimensional supergravity in the flat background.

In order to get a better grasp of the constructions from [11], it is desirable to identify Θ with a construction already known in homological algebra, e.g. complexes of DeRham, Dolbeault, tangential CR complex and such.

In this paper we construct a manifold \mathcal{P} which we call the odd twistor transform of the D=11 supergravity equations (SUGRA). This is a super CR-manifold (see [29] for an introduction to CR structures for physicists). We establish a quasi-isomorphism of Θ and the tangential CR complex of \mathcal{P} .

We emphasize that CR structures are ubiquitous in twistor theory [18] and that their appearance in our work is not surprising. What is unexpected is that our twistor transform encodes solutions of SUGRA rather than equations of conformal supergravity or equations of anti self-duality, as the conventional (ambi-)twistor constructions do (cf. [32]).

The odd twistor transform is a modification of the superspace gravity of Brink and Howe [9]. In this approach a solution of SUGRA on a Lorentz oriented

spin-manifold M^{11} is encoded by a super-extension $\mathcal{M} = \mathcal{M}^{11|32}$ and a rank $(0|32)$ subbundle

$$F \subset T(\mathcal{M}) \quad (6)$$

of the tangent bundle. The manifold $\mathcal{M} = \Pi S$ is the total space of the spinor bundle over M with the parity of fibers reversed. If the vector fields ξ_1, ξ_2 are in F , the commutator $[\xi_1, \xi_2]$ might not be. The Frobenius tensor, or the torsion $T(\xi_1, \xi_2)$, is the normal component of $[\xi_1, \xi_2]$. T is a map of vector bundles

$$T : \Lambda^2 F \rightarrow N = T(\mathcal{M})/F \quad (7)$$

Bear in mind that since F is odd, T is symmetric. By the results of [9], the equations of SUGRA can be written succinctly as

$$T_{\alpha\beta}^i = \Gamma_{\alpha\beta}^i \quad (8)$$

where $T_{\alpha\beta}^i$ is the matrix of T in suitable local bases. For example, in flat space-time $\mathbb{R}^{11|32}$ the subbundle F is spanned by vector fields (5). The odd subbundle F has symmetric inner product $C(\cdot, \cdot)$, which has a skew-symmetric matrix $C_{\alpha\beta}$. The normal bundle N carries a Lorentz metric g_{ij} . Identity $\det(\Gamma_{\alpha\beta}^i y_i) = q(y)^{16}$ where

$$q(y) = g^{ij} y_i y_j \quad (9)$$

defines the conformal class g_{ij} (see [9] for an explanation of how to fix g_{ij} in its conformal class and how to define the odd symmetric pairing $C_{\alpha\beta}$ on F).¹ In the flat case $C_{\alpha\beta}$ comes from the symplectic $\text{Spin}(10, 1, \mathbb{R})$ -invariant form on $\mathfrak{so}_{10,1}^{\mathbb{R}}$. To define $\Gamma^{\alpha, \beta, i_1, \dots, i_k}$, we use g_{ij} , $C_{\alpha\beta}$ and (8).

The *odd twistor transform* $\mathcal{P} = \mathcal{P}_{\mathcal{M}, F}$ of the SUGRA datum (\mathcal{M}, F) is a relative Isotropic Grassmannian $\text{OGr}(2, 11)_{\mathcal{M}}$. Locally as a real manifold it is a product

$$\mathbb{R}^{11|32} \times \text{OGr}(2, 11) = \mathbb{R}^{11|32} \times \text{SO}(11, \mathbb{R})/\text{U}(2) \times \text{SO}(7, \mathbb{R}). \quad (10)$$

¹In this note we shall not make a distinction between an orthonormal basis and a weak orthonormal basis in N , that is a collection of sections $f_i \in N$ such that $g(f_i, f_j) = \pm c \delta_{ij}$, $c > 0$. The last notion makes sense when the metric is defined only up to a scaling factor.

The group $\mathrm{SO}(n, \mathbb{R})$ is a compact form of $\mathrm{SO}(n) \stackrel{\text{def}}{=} \mathrm{SO}(n, \mathbb{C})$. A point in $\mathrm{OGr}(2, 11)$ is represented by a light-like (or isotropic) two-plane W in the complexified Minkowski space. More formally this can be said as follows. The complexification V^{11} of $V^{10,1} \mathbb{R}$ is equipped with the complexified inner product (\cdot, \cdot) . As an algebraic variety $\mathrm{OGr}(2, 11)$ is isomorphic to

$$\{W \subset V^{11} \mid \dim W = 2, (\cdot, \cdot)|_W = 0\}.$$

To get an equivalent description of $\mathrm{OGr}(2, 11)$, we can consider isotropic two-planes in the complexification of an Euclidean eleven-dimensional space. This is unnatural from the standpoint of the gravity theory, but explains why $\mathrm{OGr}(2, 11)$ is a coset space (10) and makes the topological structure of $\mathrm{OGr}(2, 11)$ more apparent. The superspace $\mathrm{OGr}(2, 11)_{\mathcal{M}}$ can be embedded in the projective bundle $\mathbf{P}(\Lambda^2 N^{\mathbb{C}})$ using the Plücker embedding. $\mathrm{OGr}(2, 11)_{\mathcal{M}}$ is defined fiber-wise by equations (50), which are written in a local g_{ij} -orthonormal basis of N . The space $\mathcal{P} = \mathrm{OGr}(2, 11)_{\mathcal{M}}$ has real dimension $(2 \times 15 + 11|2 \times 8 + 16)$.

The manifold \mathcal{P} has a CR-structure (Definition 4) defined by means of the complex subbundle of the complexified tangent bundle $H^{1,0} \subset T^{\mathbb{C}}(\mathcal{P})$. We begin the explanation of its construction with a remark that the fibers of the projection

$$p: \mathcal{P} \rightarrow \mathcal{M} \tag{11}$$

are complex manifolds, which are isomorphic to $\mathrm{OGr}(2, 11)$. The space $H_x^{1,0}, x \in \mathcal{P}$, is characterized by the condition that the kernel of the differential Dp

$$H_x^{1,0} \xrightarrow{Dp} T_z^{\mathbb{C}}, \quad z = p(x) \tag{12}$$

is the complex tangent space $T_x(\mathrm{OGr}(2, 11)) = T_x^{vert}$ to the fiber $p^{-1}(z)$ at x . The image $(Dp)H_x^{1,0}$ is spanned by the complex vector fields

$$\xi_{\beta} = \bar{a}^{ij} \Gamma_{\beta ij}^{\alpha} \eta_{\alpha}. \tag{13}$$

The variables a^{ij} are the Plücker coordinates (48) of the isotropic two-plane W corresponding to $x \in p^{-1}(z) \cong \mathrm{OGr}(2, 11)$, and $\{\eta_{\alpha}\}$ is a basis in F_z . Complex conjugation on $\Lambda^2 N^{\mathbb{C}}$ defines an involution ρ on \mathcal{P} .

Here is our first result about \mathcal{P} .

Proposition 1 *Let (\mathcal{M}, F) be a real $(11|32)$ -dimensional supermanifold, such that the Frobenius tensor of rank $(0|32)$ distribution F satisfies (8). Then the CR structure $H^{1,0}$ given by (12) on the relative Isotropic Grassmannian $\mathcal{P}_{\mathcal{M}, F} = \text{OGr}(2, 11)_{\mathcal{M}}$ is integrable. The complex involution ρ on $\mathcal{P}_{\mathcal{M}, F}$ maps $H^{1,0}$ to $H^{0,1}$ and $H^{1,0} \cap H^{0,1} = \{0\}$.*

See Section 2 for the proof and discussion. The inverse transform $\mathcal{P} \Rightarrow (\mathcal{M}, F)$ is defined if \mathcal{P} satisfies conditions of the following theorem (see Section 6 for details).

Proposition 2 *Let \mathcal{P} be a globally embeddable (see Definition 6) $(2 \times 15 + 11|2 \times 8 + 16)$ -dimensional super CR manifold. Suppose that \mathcal{P} satisfies conditions $(1, 2, 3, 4)$ in Section 6. Then \mathcal{P} is isomorphic to the odd twistor transform of some (\mathcal{M}, F) .*

In this Proposition conditions $(1, 2, 3, 4)$ seem to be essential conditions. It is desirable to get rid of the global embeddability because it is not intrinsic to the CR nature of the problem.

The tangential Cauchy-Riemann complex (cf. [6] and (19)) $\Omega_{H^{0,1}} = \bigoplus_{p \geq 0} \Omega_{H^{0,1}}^{0,p}$ is an analogue of the Dolbeault complex for CR (super)manifolds. A generalization $\Theta_{\mathcal{M}, F}$ of the complex (1) can be defined for a non-flat space-time \mathcal{M} and a distribution F (see Section 3 and [7], [5] for details and further development).

The map (11) induces a homomorphism of differential graded algebras

$$p_{H^{0,1}}^* : \Theta_{\mathcal{M}, F} \rightarrow \Omega_{H^{0,1}} \quad (14)$$

(see Section 3 for details). Note that $\mathcal{P}_{\mathcal{M}, F}$ has smaller dimension than the space underlying $\Theta_{\mathcal{M}, F}$. In this sense $\mathcal{P}_{\mathcal{M}, F}$ gives a more economical description of SUGRA.

Our main result is the comparison of the cohomologies of $\Theta_{\mathcal{M}, F}$ and $\Omega_{H^{0,1}}(\mathcal{P})$ (see the end of Section 5 and Section 4):

Proposition 3 *The map $p_{H^{0,1}}^*$ defines an isomorphism between the D -cohomology of $\Theta_{\mathcal{M}, F}$ and the ρ^* -real $\bar{\partial}_{H^{0,1}}$ -cohomology of $\Omega_{H^{0,1}}(\mathcal{P})$.*

Recall that D -cohomology of $\Theta_{\mathcal{M},F}$ have an interpretation of solutions of linearized equations of SUGRA. It would be interesting in light of this result to explore the possibility of a formulation of the full nonlinear equations on \mathcal{P} .

We conclude the introduction with a list of related problems.

1. The action of linearized gravity theory in the pure spinor approach [4] has the form $S = \int d^{11}x \langle \Psi Q \Psi \rangle$, where the norm $\langle \rangle$ is such that $\langle \lambda^7 \theta^9 \rangle = 1$. Proposition 3 can be interpreted as a statement about an isomorphism of the space of solutions of the equations of linearized supergravity $Q\Psi = 0$ and space of solutions of $\bar{\partial}_{H^{0,1}} f = 0, f \in \Omega_{H^{0,1}}(\mathcal{P})$. It is plausible that $p_{H^{0,1}}^*$ defines an equivalence of the actions S and $\int_{\mathcal{P}} d\mu f \bar{\partial}_{H^{0,1}} f$, where $d\mu$ is some integral volume form on \mathcal{P} derived from the norm $\langle \rangle$. It would be interesting to find $d\mu$ using ideas of [26], [22]. The next problem is closely related.
2. The work [11] gives a description of the supergravity Lagrangian $\mathcal{L}_{\text{SUGRA}}$ in a superspace formulation with auxiliary pure spinor fields. Some of the terms of $\mathcal{L}_{\text{SUGRA}}$ (such as $\Gamma_{\alpha\beta}^{ij} \lambda^\alpha \lambda^\beta$) are sections of a bundle on $\text{OGr}(2, 11)$. It is tempting to speculate that the Lagrangian can be defined on \mathcal{P} . The idea is to interpret the bracket $\{, \}$ defined by the formula

$$\Psi\{\Psi, \Psi\} \stackrel{\text{def}}{=} \lambda \Gamma_{ab} \lambda \Psi R^a \Psi R^b \Psi,$$

taken from the full supergravity Lagrangian [11] as a weak Poisson structure (a G_∞ -structure with a trace in the mathematical slang). If this guess is correct, the technique of [25] can be used to transfer the G_∞ -structure to $\Omega_{H^{0,1}}(\mathcal{P})$.

3. SUGRA is a low energy limit of M-theory. It is believed that M-theory properties are related to the supermembrane [19] [3][16]. Pure spinors play a fundamental role in the covariant formulation of the supermembrane [4]. It is interesting to translate supermembrane from the superspace to the twistor space. One of the attractive feature of twistors is that the polynomials $a^{ij} = \Gamma_{\alpha\beta}^{ij} \lambda^\alpha \lambda^\beta$ after the blowup become basic generators

[27]. The nonlinear constraint $\lambda\Gamma_{ij}\lambda\Pi_J^j = 0$ [4], where Π_J^j is the canonical momentum, simplifies to

$$a^{ij}\Pi_J^j = 0. \quad (15)$$

It would be interesting to systematically apply the odd twistor transform to the supermembrane and its double reduction - strings.

We plan to address these questions in future publications.

Here is an outline of the structure of the paper. In Section 2 we establish integrability of the CR structure of \mathcal{P} . In Section 3 we define the tangential CR complex $\Omega_{H^{0,1}}(\mathcal{P})$ and the non-flat generalization $\Theta_{\mathcal{M},F}$ of the complex (1). In the same section we also define the map $p_{H^{0,1}}^*$ between these complexes. Reality conditions, which are used later to characterize physical fields, are formulated in Section 4. It is known that not every holomorphic supermanifold admits a projection onto the underlying manifold. Supermanifolds having this property are called split. In Section 5 we define obstruction of being split in the context of CR manifolds that are odd twistor transforms. We also establish that the map $p_{H^{0,1}}^*$ defines an isomorphism on cohomology. In Section 6 we invert the odd twistor transform under certain assumptions of analyticity. Section 7 briefly describes an interesting even modification of the CR structure on \mathcal{P} . The appendices contain discussion of some technical points. In particular, in Appendix A we justify the local description of the map $p_{H^{0,1}}^*$. The Plücker embedding of $\text{OGr}(2, 11)$ is characterized by equations in Appendix B. Orbits of $\text{SO}(10, 1, \mathbb{R})$ in $\text{OGr}(2, 11)$ are listed in Appendix C. The super-Poincaré group acts on the odd twistor transform \mathcal{P} of the flat solution of SUGRA. The group preserves the CR structure and has a dense orbit in \mathcal{P} . In Appendix D we give a Lie algebraic description of the CR structure on this orbit, considered as a homogenous space.

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2 Integrability of the CR structure on $\mathcal{P}_{\mathcal{M},F}$

We devote this section to the proof of integrability of the CR structure on $\mathcal{P}_{\mathcal{M},F}$. But first we give a formal definition of the CR structure.

Definition 4 (cf. [6]) *Let Y be a C^∞ super-manifold, equipped with a subbundle $H^{1,0}$ of the complexified tangent bundle $T^\mathbb{C} = T^\mathbb{C}(Y)$. If $H^{1,0} \cap \overline{H^{1,0}} = 0$ (we shall call it a nondegeneracy condition), then Y is a Cauchy-Riemann (CR) manifold. If the space of sections in $H^{1,0}$ (or in $H^{0,1} = \overline{H^{1,0}}$) is closed under the commutator (we shall call it an involutivity condition), then the CR structure is integrable.*

Verification of the nondegeneracy condition is done in [27]. Let us check integrability of $H^{1,0}$. The vector fields ξ_β commute with the local vertical holomorphic vector fields in notations of (13). Locally we decompose the tangent bundle $T(\mathcal{M})$ into a direct sum $F + N$. With a suitable choices of local bases $\{\eta_\alpha\}$ of F and $\{v_i\}$ of N the commutators $[\eta_\alpha, \eta_\beta]$ decompose into $\eta_{\alpha\beta} + \Gamma_{\alpha\beta}^i v_i$, where $\eta_{\alpha\beta}$ are some sections of F . The commutator of the vector fields ξ_γ (13) is

$$[\xi_\gamma, \xi_\delta] = \bar{a}^{ij} \Gamma_{\gamma ij}^\alpha \left(\bar{a}^{kl} \Gamma_{\delta kl}^\beta \eta_{\alpha\beta} \right) + \bar{a}^{ij} \Gamma_{\gamma ij}^\alpha \bar{a}^{kl} \Gamma_{\delta kl}^\beta \Gamma_{\alpha\beta}^i v_i. \quad (16)$$

The N -component has coefficients

$$P^s(\bar{a}) = \Gamma_{\alpha\alpha'}^s \bar{a}^{ij} \Gamma_{\beta ij}^\alpha \bar{a}^{kl} \Gamma_{\beta' kl}^{\alpha'} \quad (17)$$

These coefficients are zero because $P^s(a)$ transforms as a $\text{SO}(11)$ vector. However a vector representation is not a subrepresentation of $\text{Sym}^2[\Lambda^2 V^{11}]^2$ [27]. The remaining terms in (16) are sections of $H^{1,0}$. This proves integrability.

The involution ρ from the introduction leaves equations that define $\text{OGr}(2, 11)$ (50) invariant. A point $W = \overline{W} \in \text{OGr}(2, 11)$ is a complexification of the light-like real plane. A set of such planes is empty in Lorentz signature. We conclude that ρ is fixed point free on \mathcal{P} . The involution turns \bar{a}^{ij} into a^{ij} in (13) and swaps $H^{1,0}$ with $H^{0,1}$.

²In this paper $\text{Sym}^i E$ stands for the i -th symmetric power of a representation or a vector bundle.

3 The complexes $\Omega_{H^{0,1}}(\mathcal{P})$ and $\Theta_{\mathcal{M},F}$

In this section we define the complexes that appeared in the introduction.

Construction of the tangential Cauchy-Riemann complex is based on the observation that the CR structure is integrable if and only if the ideal

$$I = \{\omega \in \Omega \otimes \mathbb{C} \mid \forall \xi_i \in H^{0,1} \quad \omega(\xi_1, \dots, \xi_{\deg \omega}) = 0\} = \bigoplus_{p \geq 0} I^p \subset \Omega \quad (18)$$

in the algebra differential forms $\Omega = \Omega(Y)$ is d -closed: $d(I) \subset I$ (see e.g. [6] for the proof of the even case). A CR-form $\omega \in$

$$\Omega_{H^{0,1}}^{0,p} \stackrel{\text{def}}{=} \Omega^p / I^p \quad (19)$$

is $\sum_{i_1, \dots, i_p} \omega_{i_1, \dots, i_p} \bar{\nu}^{i_1} \wedge \dots \wedge \bar{\nu}^{i_p}$ where $\bar{\nu}^i$ are complex-linear functionals on $H^{0,1}$. The tangential Cauchy-Riemann operator $\bar{\partial} = \bar{\partial}_{H^{0,1}}$ in $\Omega_{H^{0,1}} = \Omega_{H^{0,1}}(Y) = \bigoplus_{p \geq 0} \Omega_{H^{0,1}}^{0,p}(Y)$ is induced by the DeRham differential d . The map of complexes

$$res_{H^{0,1}} : \Omega \rightarrow \Omega_{H^{0,1}} \quad (20)$$

is a restriction onto $H^{0,1}$. In our applications we are mainly interested in $\Omega_{H^{0,1}}(\mathcal{P})$.

Another complex announced in the introduction is $\Theta_{\mathcal{M},F}$. It is a generalization of Θ (1). In order to define $\Theta_{\mathcal{M},F}$, we choose linearly independent even forms E^i that vanish on F . These forms are a part of the vielbein and generate a locally free subsheaf in $\Omega^1(\mathcal{M})$ of rank (11|0). Let x^i, θ^α be local coordinates on \mathcal{M} . Without a loss of generality E^i is equal to $dx^i - T_{\alpha\beta}^i(x, \theta) \theta^\alpha d\theta^\beta$. The forms E^i characterize the distribution (6). Equality

$$d(E^i) = \Gamma_{\alpha\beta}^i d\theta^\alpha d\theta^\beta + E^k G_k^i, \quad (21)$$

where G_k^i are some one-forms, is equivalent to (7,8). This implies that forms

$$E^i, \Gamma_{\alpha\beta}^i d\theta^\alpha d\theta^\beta \quad (22)$$

generate a differential ideal $J \subset \Omega(\mathcal{M})$. We define $\Theta_{\mathcal{M},F}$ to be $\Omega(\mathcal{M})/J$. Together with x^i, θ^α variables $d\theta^\alpha = \lambda^\alpha$ are local generators of $\Theta_{\mathcal{M},F}$ - the deformed

version of the algebra (1). The algebra $\Theta_{\mathcal{M},F}$ is graded by \deg_λ . When we say that $\Theta_{\mathcal{M},F}$ is a deformation of Θ we mean that locally only the differential $D = D_{\mathcal{M},F}$ in $\Theta_{\mathcal{M},F}$ gets deformed:

$$Dx^i = T_{\alpha\beta}^i \theta^\alpha \lambda^\beta. \quad (23)$$

There is an analogue of the map (20) for $\Theta_{\mathcal{M},F}$:

$$res_F : \Omega(\mathcal{M}) \rightarrow \Omega(\mathcal{M})/J = \Theta_{\mathcal{M},F}$$

Construction of the map $p_{H_{0,1}}^*$ (14) requires a clarification. In order to define $p_{H_{0,1}}^*(\omega)$, we pick $\tilde{\omega} \in \Omega(\mathcal{M})$ such that $res_F \tilde{\omega} = \omega$. We define $p_{H_{0,1}}^*(\omega)$ to be $res_{H^{0,1}} p^* \tilde{\omega}$. This is not the end of the story. We need to verify that $p^* J \subset I$. We check this on the generators (22). It follows immediately from the definition of $H^{0,1}$ (12) that $p^* E^i$ vanishes on $H^{0,1}$. As a result $p^* E^i \in I$. The identity

$$\xi_{\delta_1} \lrcorner (\xi_{\delta_2} \lrcorner p^* \Gamma_{\alpha\beta}^i d\theta^\alpha d\theta^\beta) = 0$$

for the vector fields ξ_{δ_i} (13) is true because the polynomials (17) are zero. We conclude that $p^* \Gamma_{\alpha\beta}^i d\theta^\alpha d\theta^\beta \in I$ and $p^* J \subset I$. It implies that (14) is well defined and $p_{H_{0,1}}^*$ is a map of complexes.

Our next goal is to write $p_{H_{0,1}}^*$ in local coordinates on \mathcal{M} and \mathcal{P} . Let $a^{ij}(W)$ be the Plücker coordinates of $W \in \text{OGr}(2, 11)$ (see Appendix B). The family of vector spaces

$$\mathfrak{s}_W = \{a^{ij}(W) \Gamma_{\beta ij}^\alpha \eta_\alpha | \eta_\alpha \in \mathfrak{s}_{11}\} \subset \mathfrak{s}_{11} \stackrel{\text{def}}{=} \mathfrak{s}_{10,1} \otimes \mathbb{R} \otimes \mathbb{C} \quad (24)$$

defines a complex vector bundle

$$\mathfrak{s}_{\text{OGr}(2,11)} = \{(W, \xi) | W \in \text{OGr}(2, 11), \xi \in \mathfrak{s}_W\} \quad (25)$$

We are going to define coordinates on the total space of $\mathfrak{s}_{\text{OGr}(2,11)}$ that will be used in the local description of $p_{H_{0,1}}^*$. For this purpose, we need a basis in the space of local sections of $\mathfrak{s}_{\text{OGr}(2,11)}$. Such a basis can be seen rather explicitly. We fix a basis $\{\eta_\alpha\}$ in \mathfrak{s}_{11} that is compatible with the decomposition (43), such that $\eta_1, \dots, \eta_8 \in s^1$, $\eta_9, \dots, \eta_{24} \in s^0$, and $\eta_{25}, \dots, \eta_{32} \in s^{-1}$. We pick a plane

$W \in \text{OGr}(2, 11)$ the same as in the proof of the isomorphism 47 and choose it to be close to U in (40).

We pick kl such that $a^{kl}(U) \neq 0$. We set

$$\mu_\beta = \frac{a^{ij}(W)}{a^{kl}(W)} \Gamma_{\beta ij}^\alpha \eta_{\alpha, \beta} = 25, \dots, 32$$

Note that when $W = U$ then $\mu_{24+\alpha}$ is proportional to $\eta_{24+\alpha}$. (26)

This means that $\{\mu_\beta\}$ are linearly independent sections of $\mathfrak{s}_{\text{OGr}(2, 11)}$ in a Zariski neighborhood of U . Let $\mu^A, A = 1, \dots, 8$ be sections of the dual bundle such that

$$\mu^A(\mu_{24+B}) = \delta_B^A. \quad (27)$$

A variable λ^α defines a linear function on fibers of $\mathfrak{s}_{\text{OGr}(2, 11)}$ because the fibers are subspaces in \mathfrak{s}_{11} . It follows immediately that

$$\lambda^\alpha = \sum_{A=1}^8 \frac{a^{ij}(W)}{a^{kl}(W)} \Gamma_{24+A ij}^\alpha \mu^A \quad (28)$$

The locally defined CR-forms on $\mathcal{P}_{\mathcal{M}}$ are functions in

$$x^i, \theta^\alpha, a^{ij}, \bar{a}^{ij}, \mu^A, d\bar{a}^{ij} \quad (29)$$

that have the total $\text{GL}(2, \mathbb{C})$ -scaling degree zero(49) in a^{ij} , \bar{a}^{ij} and \bar{a}^{ij} . The map (14) keeps x^i, θ^α unchanged and replaces λ^α with the RHS of the formula (28).

We want to finish this section with a question. In general the Poincaré lemma fails in a tangential CR complex (see e.g.[6]). Does it fail in $\Omega_{H^{0,1}}(\mathcal{P}_{\mathcal{M}, F})$?

4 Reality conditions

The classical 11-D supergravity is defined over the field of real numbers, whereas we work over the complex numbers. The missing reality conditions will be formulated in this section.

A real analytic function $f(z) = \sum_{k=0}^\infty c_k z^k, c_k \in \mathbb{R}$ is characterized by the identity $f(z) = \overline{f(\bar{z})}$. More generally, a real analytic function f on a complex

manifold X equipped with an anti-holomorphic involution ρ is characterized by

$$f(z) = \overline{f(\rho(z))}.$$

This definition of reality extends to the space of complex smooth differential forms $\Omega^k = \bigoplus_{i+j=k} \Omega^{i,j} = \bigoplus_{i+j=k} \Omega^{i,j}(X)$. The involution ρ maps $\omega \in \Omega^{i,j}$ to $\rho^* \omega \in \Omega^{j,i}$. Bear in mind that $\overline{\rho^* \omega} \in \Omega^{i,j}$ and $\overline{\rho^* \bar{\partial}} = \bar{\partial}$. A real form satisfies

$$\omega = \overline{\rho^* \omega}$$

Real forms define a sub-complex in $\bigoplus_j \Omega^{i,j}$. The definition extends to super CR manifolds: a map ρ is a C^∞ CR involution if $\rho^* H^{1,0} \subset \overline{H^{1,0}}$ and $\rho^2 = \text{id}$. The role of $(\Omega^{0,p}, \bar{\partial})$ is played by the tangential CR complex, in which ρ defines an anti-linear automorphism of $\Omega_{H^{0,1}}^{0,p}$.

5 A cohomological invariant of the CR structure on \mathcal{P}

In this section we develop rudiments of the structure theory of super CR manifolds adapted to the odd twistor space \mathcal{P} . The structure theory of holomorphic supermanifolds was studied in [24]. A holomorphic $(n|m)$ -dimensional supermanifold Y has the following basic invariants: the underlying even n -dimensional manifold Y_{red} and a holomorphic rank m vector bundle \mathcal{G} . A more refined invariant is a sequence of characteristic classes ω_i , with the simplest $\omega_1 \in H^1(Y_{\text{red}}, \Lambda^2 \mathcal{G}^* \otimes T(Y_{\text{red}}))$. Keep in mind that these characteristic classes have no immediate relation to the topological characteristic classes of vector bundles. The manifold Y can be thought of as a deformation of the split manifold $Y_{\text{split}} = \Pi \mathcal{G}$, the deformation is trivial on Y_{red} . The characteristic class ω_1 in $H^1(Y_{\text{split}}, T(Y_{\text{split}}))$ (we interpret sections of $\Lambda^2 \mathcal{G}^* \otimes T(Y_{\text{red}})$ as local vector fields on $\Pi \mathcal{G}$) is zero when $Y \cong Y_{\text{split}}$. A non zero ω_1 is an obstruction to splitting of Y . The Čech approach to cohomology was used in [24] for the construction of ω_1 . Dolbeault cohomology has the same basic functionality, but it is more

flexible because it admits a generalization to the CR case. We shall not attempt to develop a theory of characteristic classes of super CR manifolds in the full generality. Instead, the goal of this section is to identify the cocycle ω_1 and the group it belongs to in the case of $\mathcal{P}_{\mathcal{M},F}$.

In our definition of $\omega_1(\mathcal{P})$, we certainly want to follow the structure theory of holomorphic supermanifolds outline above. Obviously, \mathcal{P}_{red} is a relative Isotropic Grassmannian $\mathcal{P}_{\text{red}} \cong \text{OGr}(2, 11)_M$ with the projection $\text{OGr}(2, 11)_M \xrightarrow{p} M$. The split form $\mathcal{P}_{\text{split}} \cong \Pi p^* S_M$ has a CR-structure that is nontrivial only on the fibers of the projection $q_{\text{split}} : \mathcal{P}_{\text{split}} \rightarrow M$. We denote a fiber by $\widetilde{\text{OGr}}(2, 11)$. Then

$$\widetilde{\text{OGr}}(2, 11) \cong \text{OGr}(2, 11) \times \Pi \mathfrak{s}_{10,1} \mathbb{R} \quad (30)$$

The subbundle $H^{1,0} \subset T^{\mathbb{C}}(\widetilde{\text{OGr}}(2, 11))$ is still defined by formulas (12,13) where $\{\eta_\alpha\} \subset \Pi \mathfrak{s}_{11}$ is a basis in the space of the constant spinors. We shall define now a collection of cocycles $\{\gamma^i\}$ that are tangential CR forms over $\widetilde{\text{OGr}}(2, 11)_b \cong q_{\text{split}}^{-1}(b)$.

The construction of $\{\gamma^i\}$ simplifies if we present $\bar{\partial}_{H^{0,1}}$ as a sum of two anti-commuting differentials d_I and d_{II} . The differential $\bar{\partial}_{H^{0,1}}$ has the bi-degree $(1, 1)$ with respect to the bigrading (c, c') on $\Omega_{H^{0,1}}(\mathcal{P})$ defined by the rule

$$(c, c') = (\deg_{d\bar{a}} f, \deg_\mu f), f \in \Omega_{H^{0,1}}(\mathcal{P}).$$

The $(1, 0)$ component of $\bar{\partial}_{H^{0,1}}$ is $d_I = d\bar{a}^{ij} \frac{\partial}{\partial \bar{a}^{ij}}$. The $(0, 1)$ component is

$$d_{II} = \mu^A \left(h(x, \theta, a)_A^\alpha \frac{\partial}{\partial \theta^\alpha} + g(x, \theta, a)_A^i \frac{\partial}{\partial x^i} \right) \quad (31)$$

We need to describe local sections of $\Omega_{H^{0,1}}(\widetilde{\text{OGr}}(2, 11)_b)$ in a more down-to-earth terms. If we set x^i to constants b^i ($b = (b^i)$), then the remaining variables (29) by definition are (possibly singular) sections of $\Omega_{H^{0,1}}(\widetilde{\text{OGr}}(2, 11)_b)$ (30). The space of C^∞ sections of $\Omega_{H^{0,1}}(\widetilde{\text{OGr}}(2, 11)_b)$ is isomorphic to the space of sections of

$$\bigoplus_{p,i,j \geq 0} \Omega^{0,p} \Lambda^i \mathfrak{s}_{11}^* \otimes \text{Sym}^j \mathfrak{s}_{\text{OGr}(2,11)}^*.$$

Bear in mind that a local section of $\text{Sym}^j \mathfrak{s}_{\text{OGr}(2,11)}^*$ is a local holomorphic function on the total space of $\mathfrak{s}_{\text{OGr}(2,11)}$ of degree i homogeneity in the fiber-wise direction (see Section 3 for details). The differential d_I acts on the elements of the algebra generated by $\theta^\alpha, a^{ij}, \bar{a}^{ij}, \mu^A, d\bar{a}^{ij}$. In general $d_{II}f(\theta^\alpha, a^{ij}, \bar{a}^{ij}, \mu^A, d\bar{a}^{ij})$ is x -dependent, but if we remove all terms in d_{II} (31) of degree one and higher in θ , the remaining differentiation d'_{II} transforms $\Omega_{H^{0,1}}(\widetilde{\text{OGr}}(2,11))$ to itself and squares to zero. By definition $\bar{\partial}_{H^{0,1}, \widetilde{\text{OGr}}(2,11)} = d_I + d'_{II}$.

We are ready to describe the cocycles γ^i in local coordinates. In the flat case $\gamma^i = \mu^A g(\theta, b)_A^i = \bar{\partial}_{H^{0,1}} x^i = \Gamma_{\alpha\beta}^i \theta^\alpha \lambda^\beta, i = 1, \dots, 11$, with λ^β replaced by (28). Elements $\gamma^i(b)$ are sections of $\Omega_{H^{0,1}}(\widetilde{\text{OGr}}(2,11)_b)$. They can be packaged into a single object $\omega_1(\mathcal{P}) = \gamma^i(x) \frac{\partial}{\partial x^i} \in \Omega_{H^{0,1}}(\mathcal{P}_{\text{split}}, T_{CR}(\mathcal{P}_{\text{split}}))$ in which x -dependence is restored. Here $T_{CR}(\mathcal{P}_{\text{split}})$ is $T^{\mathbb{C}}(\mathcal{P}_{\text{split}})/H^{0,1}$. The first cohomology group of $\Omega_{H^{0,1}}(\mathcal{P}_{\text{split}}, T_{CR}(\mathcal{P}_{\text{split}}))$ is an analogue of $H^1(Y_{\text{split}}, T(Y_{\text{split}}))$ in the holomorphic theory. In the non flat case

$$\gamma^i \text{ is the leading term of } \bar{\partial}_{H^{0,1}} x^i \text{ in } \theta \text{ of degree } \deg_\theta = 1. \quad (32)$$

Equation

$$(d_I + d'_{II})\gamma^i = 0 \quad (33)$$

follows from $\bar{\partial}_{H^{0,1}}^2 = 0$. Some simple properties of γ^i are established in [27]. In particular [27] contains a computation of the cohomology of $\Omega_{H^{0,1}}(\widetilde{\text{OGr}}(2,11))$. Elements γ^i generate $H^1(\Omega_{H^{0,1}}(\widetilde{\text{OGr}}(2,11))) \cong V^{11}$. They transform covariantly as vectors under the action of $\text{Spin}(10,1, \mathbb{R})$.

Note that if $\bar{\partial}_{H^{0,1}} x^i$ were all zero, the manifold \mathcal{P} would be split. The manifold \mathcal{P} would still be split were the elements $\bar{\partial}_{H^{0,1}} x^i$ just cohomologous to zero in an imprecise sense, which takes into account the local coordinate change. This is why ω_1 is an obstruction to splitting of \mathcal{P} .

The proof of Proposition 3 is simple, provided we take for granted the following result.

Proposition 5 (cf.[27])

$$H^0(\text{OGr}(2,11), \text{Sym}^i \mathfrak{s}_{\text{OGr}(2,11)}^*) = A_i,$$

$$H^k(\text{OGr}(2, 11), \text{Sym}^i \mathfrak{s}_{\text{OGr}(2, 11)}^*) = 0, k \geq 1.$$

The computation of $H^i(\Omega_{H^{0,1}})$ can be done in two stages. The first is the computation of the d_I -cohomology. The resulting algebra \mathcal{E} has less generators than $\Omega_{H^{0,1}}$. The second is the computation of the d_{II} -cohomology of \mathcal{E} . Notice that d_I does not depend on x and θ . Elements of the algebra generated by $\mu, a, \bar{a}, d\bar{a}$ are possibly singular C^∞ Dolbeault forms with values in $\bigoplus_{n \geq 0} \text{Sym}^n \mathfrak{s}_{\text{OGr}(2, 11)}^*$. The d_I -cohomology of the subalgebra of regular C^∞ forms is the algebra A (2) (see Proposition 5). This explains why the stage two of the procedure is identical to the D -cohomology computation in $\Theta_{\mathcal{M}, F}$. The reality condition enforced by ρ picks up polynomials in the generators of A with real coefficients.

The CR structure on \mathcal{P} is compatible with the trivial CR structure on the relative grassmannian $\text{OGr}(2, 11)_M = \mathcal{P}_{\text{red}}$. Let J be the kernel of the restriction map $\Omega_{H^{0,1}}(\mathcal{P}) \rightarrow \Omega_{H^{0,1}}(\mathcal{P}_{\text{red}})$. The arguments from the previous paragraph become global in the framework of the spectral sequence associated with the filtration $J^{\times n}$ in $\Omega_{H^{0,1}}(\mathcal{P})$. The spectral sequence degenerates in the E^2 term as a consequence of Proposition 5.

6 The inverse transform

The odd twistor transform converts a SUGRA datum (\mathcal{M}, F) that satisfies (7, 8) into a $(2 \times 15 + 11|2 \times 8 + 16)$ -dimensional CR manifold $\mathcal{P} = \mathcal{P}_{\mathcal{M}, F}$. In this section we shall concern ourself with the intrinsic characterization of \mathcal{P} in the class $(2 \times 15 + 11|2 \times 8 + 16)$ -dimensional CR manifolds. To summarize our previous discussion, we list the most important characteristics of \mathcal{P} :

1. The complexified tangent bundle $T^{\mathbb{C}}(\mathcal{P})$ contains a complex rank $(15|8)$ subbundle $H^{1,0}$ that defines an integrable CR-structure.
2. There is a non-empty family $\text{OGr}(2, 11) \subset \mathcal{P}$ of CR-holomorphic Orthogonal Grassmannians. The real normal bundle is trivial $N_{\text{OGr}(2, 11)} \cong \text{OGr}(2, 11) \times V^{10,1} \otimes \mathbb{R} \times \Pi \mathfrak{s}_{10,1}^{\mathbb{R}}$. The bundle $H^{1,0}|_{\text{OGr}(2, 11)}$ is a (trivial) ex-

tension

$$0 \rightarrow T(\text{OGr}(2, 11)) \rightarrow H^{1,0} \rightarrow \Pi \bar{\mathfrak{S}}_{\text{OGr}(2,11)} \rightarrow 0 \quad (34)$$

3. The preimage of $\text{OGr}(2, 11) \subset \mathcal{P}_{\text{red}}$ in $\mathcal{P}_{\text{split}}$ as a CR manifold is isomorphic to $\widetilde{\text{OGr}}(2, 11)$ (30).

Let $x^i, i = 1, \dots, 11$ be local even independent functions on \mathcal{M} . By abuse of notations, we denote p^*x^i by x^i . In the notations of Section 3, the differential $\bar{\partial}_{H^{0,1}}x^i \in \Omega_{H^{0,1}}^1(\mathcal{P})$ can locally be written as

$$g_A^i(x, \theta, a)\mu^A. \quad (35)$$

Keep in mind that it is automatically independent of \bar{a} and $d\bar{a}$. A section $\gamma_{\alpha,A}^i(b, a)\theta^\alpha\mu^A$ of $\Omega_{H^{0,1}}^1(\widetilde{\text{OGr}}(2, 11)_b)$ is the leading in θ term of (35). The classes of γ^i define a basis in $H^1(\Omega_{H^{0,1}}(\widetilde{\text{OGr}}(2, 11)))$ (cf. discussion in Section 5).

4. There is a fixed point free involution $\rho: \mathcal{P} \rightarrow \mathcal{P}$ that maps $H^{1,0}$ to $H^{0,1}$. At least one of $\text{OGr}(2, 11)$ in the family is ρ -invariant. The involution commutes with a holomorphic $\text{SO}(10, 1, \mathbb{R})$ action on the $\text{OGr}(2, 11)$.

Definition 6 *A supermanifold \mathcal{P} is globally embeddable if it is a closed CR submanifold of a complex $(26|24)$ -dimensional manifold $\mathcal{P}_{\mathbb{C}}$. We assume that ρ extends to $\mathcal{P}_{\mathbb{C}}$ as fixed-point free antiholomorphic involution.*

Global embeddability of real-analytic CR structures on ordinary manifolds was established by Andreotti and Fredricks [1]. Presumably, their technique admits a super-extension that can be applied to a real-analytic \mathcal{P} . Meanwhile we just simply assume that \mathcal{P} is globally embeddable.

We shall describe how to construct a super space-time \mathcal{M} from $\mathcal{P} \subset \mathcal{P}_{\mathbb{C}}$ that satisfies conditions (1,2,3,4). The idea goes back to Penrose. We identify \mathcal{M} with the ρ -real points in the moduli space $\mathcal{M}^{\mathbb{C}}$ of $\text{OGr}(2, 11) \subset \mathcal{P}_{\mathbb{C}}$.

The existence theorem for the versal family of compact super subvarieties [30] relies on vanishing of the cohomology groups associated with the normal bundle

of the subvariety. We begin with a computation of these groups. It follows from (34) and Assumption (2) that the holomorphic normal bundle $N_{\text{OGr}(2,11)}$ is the quotient of $\text{OGr}(2,11) \times V^{11} \times \Pi\mathfrak{s}_{11}$ by $\Pi\mathfrak{s}_{\text{OGr}(2,11)}$ (25). The formal tangent space to the moduli of deformations $\text{OGr}(2,11) \subset \mathcal{P}_{\mathbb{C}}$ is isomorphic to $H^0(N_{\text{OGr}(2,11)}) = H^0(\text{OGr}(2,11), N_{\text{OGr}(2,11)})$. The space of obstructions is $H^1(N_{\text{OGr}(2,11)})$ (cf.[20]).

Proposition 7 *Let Y be a compact projective homogeneous space of a complex semisimple group G , with the Lie algebra \mathfrak{g} . The nontrivial cohomology of the structure sheaf and the tangent sheaf are $H^0(Y, \mathcal{O}) = \mathbb{C}$, $H^0(Y, T(Y)) = \mathfrak{g}$ respectively.*

Proof. The proof follows from Theorem VII in [8]. ■

This verifies that $H^1(\text{OGr}(2,11), T(\text{OGr}(2,11))) = \{0\}$ and that the space $\text{OGr}(2,11)$ is rigid [21].

The nonzero even component of the cohomology is

$$H^0(\text{OGr}(2,11), N)^{\text{even}} = V^{11}$$

(Proposition 7). We extract the odd part from the long exact sequence:

$$\begin{aligned} 0 \rightarrow H^0(\Pi\mathfrak{s}_{\text{OGr}(2,11)}) \rightarrow H^0(\mathcal{O}) \otimes \Pi\mathfrak{s}_{11} \rightarrow H^0(N_{\text{OGr}(2,11)})^{\text{odd}} \rightarrow H^1(\Pi\mathfrak{s}_{\text{OGr}(2,11)}) \rightarrow \\ \rightarrow H^1(\mathcal{O}) \otimes \Pi\mathfrak{s}_{11} \rightarrow H^1(N_{\text{OGr}(2,11)})^{\text{odd}} \rightarrow H^2(\Pi\mathfrak{s}_{\text{OGr}(2,11)}) \rightarrow \dots \end{aligned}$$

Vanishing of $H^i(\mathfrak{s}_{\text{OGr}(2,11)})$ was verified in [27]. We derive that

$$H^0(N) \cong V^{11} + \Pi\mathfrak{s}_{11}, \quad H^1(N) = \{0\} \quad (36)$$

By the super version of the Kodaira theory of deformation of compact immersions [30] $\text{OGr}(2,11) \subset \mathcal{P}^{\mathbb{C}}$ can be included in a complex-analytic versal family $\mathcal{M}^{\mathbb{C}}$. We define \mathcal{M} to be the real locus of the involution ρ in $\mathcal{M}^{\mathbb{C}}$. Note that the condition (4) on ρ is rigid [31]. In principle, \mathcal{M} might have several connected components in $\mathcal{M}^{\mathbb{C}}$. We do not reject a possibility of a disconnected $\mathcal{M}^{\mathbb{C}}$ either.

Our next task is to define the distribution F (6) that satisfies (8). We construct it using the graph of the universal family $\mathcal{Q} \subset \mathcal{P}^{\mathbb{C}} \times \mathcal{M}^{\mathbb{C}}$ (cf. [24]). It

fits into the diagram

$$\mathcal{P}^{\mathbb{C}} \xleftarrow{r} \mathcal{Q} \xrightarrow{p} \mathcal{M}^{\mathbb{C}}$$

Leaves of r are purely odd affine spaces. A fiber $r^{-1}(x)$ $x \in \text{OGr}(2, 11) \subset \mathcal{P}^{\mathbb{C}}$ is modeled by a subspace of sections of $N_{\text{OGr}(2, 11)}$ that vanish at x . The r -vertical tangent subspaces in $T(\mathcal{Q})$ under projection p span odd subbundle $F \subset T(\mathcal{M}^{\mathbb{C}})$. By the construction $p(r^{-1}(x))$ is tangential to F . On the general grounds $p^{-1}(\mathcal{M})$ is a CR submanifold in \mathcal{Q} and $r : p^{-1}(\mathcal{M}) \rightarrow \mathcal{P}^{\mathbb{C}}$ is a CR map. The complexified real tangent bundle $T^{\mathbb{C}}(p^{-1}(\mathcal{M}))$ is isomorphic to the extension of $T(\mathcal{Q})|_{p^{-1}(\mathcal{M})}$ by the antiholomorphic relative tangent bundle $\overline{T}_{\mathcal{Q}/\mathcal{M}^{\mathbb{C}}}|_{p^{-1}(\mathcal{M})}$. We assume that

$$p^{-1}(\mathcal{M}) \xrightarrow{r} \mathcal{P} \subset \mathcal{P}^{\mathbb{C}} \quad (37)$$

is a global isomorphism. Near a real point $b \in \mathcal{M}$ represented by $\text{OGr}(2, 11) \subset \mathcal{P}$ (Condition 2) a local isomorphism follows from the inverse function theorem. The subbundle $H^{0,1} \subset T^{\mathbb{C}}(p^{-1}(\mathcal{M}))$ is isomorphic to the extension of $T_{\mathcal{Q}/\mathcal{P}^{\mathbb{C}}}|_{p^{-1}(\mathcal{M})}$ by $\overline{T}_{\mathcal{Q}/\mathcal{M}^{\mathbb{C}}}|_{p^{-1}(\mathcal{M})}$. The extension is isomorphic (by the assumption (3)) to (34) and is compatible with the isomorphism (37).

It remains to verify the torsion equation (8). Let us choose even independent local holomorphic functions $z^i, i = 1, \dots, 11$ that vanish at $z \in \mathcal{M} \subset \mathcal{M}^{\mathbb{C}}$. We set $x^i = \text{Re} z^i$ and choose a vector $\xi \in T_w(\mathcal{Q})$ $w \in p^{-1}(z)$ that is tangential to a fiber of r . A vector $(Dp)\xi \in T_z(\mathcal{M}^{\mathbb{C}})$ is the image of ξ under the differential Dp of the map p . Then

$$p^* \frac{\partial x^i}{\partial (Dp)\xi} = \frac{\partial p^* x^i}{\partial \xi} = \xi \lrcorner dp^* x^i = \xi \lrcorner \bar{\partial} p^* x^i = \xi \lrcorner \bar{\partial}_{H^{0,1}} p^* x^i = \xi \lrcorner g^i = p^*((Dp)\xi \lrcorner p_* g^i).$$

The term g^i is the same as in 31. In short

$$\frac{\partial x^i}{\partial (Dp)\xi} - (Dp)\xi \lrcorner p_* g^i = 0. \quad (38)$$

The vector ξ is a value of a holomorphic vector field that is defined in a small neighborhood of w . It commutes with any local, antiholomorphic tangential to fibers of p vector field $\bar{\zeta}$. This implies that $0 = \frac{\partial}{\partial \xi} \frac{\partial p^* x^i}{\partial \bar{\zeta}} = \frac{\partial}{\partial \bar{\zeta}} \frac{\partial p^* x^i}{\partial \xi}$. From this

we conclude that expression $g^i = g_A^i(x, \theta, a)\mu^A$ (32) is a global holomorphic, x and θ -dependent section of $\mathfrak{s}_{\text{OGr}(2,11)}^*$ - the bundle dual to $\mathfrak{s}_{\text{OGr}(2,11)}$. With the help of Proposition 5 we convert g^i to $g_{A,\alpha}^i(x, \theta)\lambda^\alpha = g_{A,\alpha}^i(x, \theta)d\theta^\alpha$ (cf. discussion in Section 3). Equation 38 is equivalent to $Dp(\xi) \lrcorner E^i = 0$ for the forms $E^i = dx^i - g_{A,\alpha}^i(x, \theta)d\theta^\alpha$. The span $\langle CX \rangle$ of the image CX of the map $(W, \xi) \rightarrow \xi \in \mathfrak{s}_{11}, (W, \xi) \in \mathfrak{s}_{\text{OGr}(2,11)}$ (see equation (25) notations) coincides with \mathfrak{s}_{11} . It implies that $\eta \lrcorner E^i = 0$ for all $\eta \in F_z$. The map p is a submersion with purely even fibers, therefore, $\dim F_z = (0, 32)$. We conclude that the independent forms $E^i, i = 1, \dots, 11$ define F_z .

By definition $\bar{\partial}_{H^{0,1}}^2 p^* x^i = \bar{\partial}_{H^{0,1}} g^i = 0$. This implies that the restriction of two-form dE^i on $CX \subset F_z$ is zero. It is known that CX is the space of complex solutions of the equations $v^i = 0$ (3) (see [27] for details). So $dE^i = c_j^i \Gamma_{\alpha\beta}^j d\theta^\alpha d\theta^\beta + G_j^i E^j$ where G_j^i are some one-forms. The proof follows if we prove that c_j^i is invertible. Invertibility follows from the conceptually simple homological considerations. We shall be sketchy and leave to the reader to fill in the missing details. First we show that the complex $\Omega_{H^{0,1}}(\widetilde{\text{OGr}}(2,11))$ computes $\text{Tor}^{\text{Sym}[\mathfrak{s}_{11}]}(A, \mathbb{C})$. By Assumption (3) classes γ^i (leading $\deg_\theta = 1$ terms of g^i (32)) define a basis in $\text{Tor}_1^{\text{Sym}[\mathfrak{s}_{11}]}(A, \mathbb{C})$. Then we compute the same group using the minimal free $\text{Sym}[\mathfrak{s}_{11}]$ resolution of A . We interpret $\gamma'^i = c_j^i \Gamma_{\alpha\beta}^j d\theta^\alpha d\theta^\beta$ as cocycles in the minimal resolution approach. To show that the classes γ'^i and γ^i coincide in $\text{Tor}_1^{\text{Sym}[\mathfrak{s}_{11}]}(A, \mathbb{C})$ and the matrix c_j^i is invertible, we use equivalence of the two approaches.

The inverse odd twistor transform of \mathcal{P} provides us with the SUGRA datum (\mathcal{M}, F) . Note that all the steps of the inverse transform are reversible and the direct transform $\mathcal{P}_{\mathcal{M}, F}$ is identically equal to \mathcal{P} .

In order to construct a versal family of $\text{OGr}(2,11)$ in \mathcal{P} that does not have a global complex embedding, more advanced analytic methods are needed.

7 An even modification of the CR structure on \mathcal{P}

There is an interesting modification of the CR structure on $\text{OGr}(2, 11)_{\mathcal{M}}$ evoked by equation (15). The modified complex distribution $H'^{1,0} \subset T^{\mathbb{C}}(\text{OGr}(2, 11)_{\mathcal{M}})$ also fits into the diagram (12). The map Dp has the same kernel. Choose a splitting $T_z^{\mathbb{C}}(\mathcal{M}) = F_z^{\mathbb{C}} + N_z^{\mathbb{C}}$. The space $(Dp)H_x'^{1,0}$ in addition to (13) contains a span of

$$v^i = \bar{a}^{ij} f_j \quad (39)$$

Vectors $\{f_j\}$ define an orthonormal basis in N_z . Note that $H'^{1,0}$ does not depend on the splitting. The proof of integrability of the new CR structure goes through. We denote the resulting CR manifold by $\mathcal{R}_{\mathcal{M},F}$. The manifold \mathcal{R} has dimension $(2 \times 17 + 7|2 \times 8 + 16)$. We plan to investigate relation of $\mathcal{R}_{\mathcal{M},F}$ to $\mathcal{P}_{\mathcal{M},F}$ and to study homological properties of $\Omega_{H^{0,1}}^{0,i}(\mathcal{R}_{\mathcal{M},F})$ in the following publications.

Appendix

A Useful decompositions of adjoint and spinor representations

The local coordinates $\mu^A(27)$, a^{ij} on the total space of the vector bundle $\mathfrak{s}_{\text{OGr}(2,11)}(25)$ depend on the choice of a base point $U \in \text{OGr}(2, 11)$. In this section we shall elaborate in this. The coordinates and the base point depend on the direct sum decomposition

$$V^{11} \cong V^7 + V^4 = V^7 + U + U' \quad (40)$$

of the fundamental vector representation of the complex $\text{SO}(11)$. V^i stands for an i -dimensional complex Euclidean space; $V^7 \perp V^4$. The two-dimensional spaces $U, U' \subset V^4$ such that $U \cap U' = \{0\}$ are isotropic. The inner product

defines a non-degenerate pairing between U and U' . The summands in (40) are irreducible $\mathrm{SO}(7) \times \mathrm{GL}(2)$ representations.

By utilizing (40) we immediately arrive at the decomposition of $\mathrm{AdSO}(11) \cong \Lambda^2 V^{11}$. it has a form of a spectral decomposition by the eigen subspaces of the central element $c \in \mathfrak{gl}_2 \cong U \otimes U^* \cong U \otimes U'$:

$$\begin{aligned} \mathrm{Ad}(\mathfrak{so}_{11})_2 &= \Lambda^2 U' \\ \mathrm{Ad}(\mathfrak{so}_{11})_1 &= V^7 \otimes U' \\ \mathrm{Ad}(\mathfrak{so}_{11})_0 &= \Lambda^2 V^7 + U \otimes U' \\ \mathrm{Ad}(\mathfrak{so}_{11})_{-1} &= V^7 \otimes U \\ \mathrm{Ad}(\mathfrak{so}_{11})_{-2} &= \Lambda^2 U \end{aligned} \tag{41}$$

The one-dimensional linear space $\Lambda^2 U$ is the Plücker image in $\mathbf{P}(\mathfrak{s}_{11})$ of the already mentioned base point $U \in \mathrm{OGr}(2, 11)$. The construction of the coordinates μ^A relied also on a direct sum decomposition of the spinor representation. We shall see now how this comes about. The space of (Dirac) spinors \mathfrak{s}_{11} is an irreducible module over the Clifford algebra $Cl(V^{11})$. The spinor representation \mathfrak{s}_{11} is symplectic [14]. Let C be the corresponding skew-symmetric $\mathrm{Spin}(11)$ -invariant inner product with a matrix $C_{\alpha\beta}$ in the basis $\{\eta_\alpha\} \subset \mathfrak{s}_{11}$. The decomposition (40) explains identifications

$$Cl(V^{11}) \cong Cl(V^7) \otimes Cl(V^4)$$

$$\mathfrak{s}_{11} \cong \mathfrak{s}_7 \otimes \mathfrak{s}_4, \dim_{\mathbb{C}} \mathfrak{s}_{11} = 32, \dim_{\mathbb{C}} \mathfrak{s}_7 = 8, \dim_{\mathbb{C}} \mathfrak{s}_4 = 4$$

The complex spinor representation \mathfrak{s}_4 is a direct sum $W_l + W_r$ of irreducible two-dimensional representations of $\mathrm{Spin}(4) \cong \mathrm{SL}(2) \times \mathrm{SL}(2)$. We arrived at a $\mathrm{Spin}(7) \times \mathrm{Spin}(4)$ -isomorphism

$$\mathfrak{s}_{11} = \mathfrak{s}_7 \otimes W_l + \mathfrak{s}_7 \otimes W_r \tag{42}$$

The spinor representation \mathfrak{s}_{11} can be further decomposed into eigenspaces of the central element $c \in \mathfrak{gl}_2$.

$$\begin{aligned}
s^1 &= \mathfrak{s}_7 \otimes f^+ \\
s^0 &= \mathfrak{s}_7 \otimes W_l \\
s^{-1} &= \mathfrak{s}_7 \otimes f^-
\end{aligned} \tag{43}$$

where f^\pm are the c -eigenvectors in W_r . It is this decomposition that is used for constructing coordinates μ^A on $\mathfrak{s}_{\text{OGr}(2,11)}$ in Section 3.

We want to verify statement (26). For this we need a description of \mathfrak{s}_{11} in terms of the Grassmann algebra [12]. Let P and P' be five-dimensional isotropic subspaces in V^{11} such that $P \cap P' = 0$. Then

$$V^{11} = P + P' + V^1, P + P' = V^{10}, V^{10} \perp V^1 \tag{44}$$

The bilinear form (\cdot, \cdot) defines a pairing between P and P' . The group $\text{GL}(5)$ acts on $P + P'$ preserving (\cdot, \cdot) . It acts trivially on V^1 . We interpret this action as an embedding

$$\text{GL}(5) \subset \text{SO}(10) \subset \text{SO}(11). \tag{45}$$

The spinor representation \mathfrak{s}_{11} , when it is restricted on the double cover $\widetilde{\text{GL}}(5)$, is isomorphic to $\Lambda(P') \otimes \det^{\frac{1}{2}}$ (see [10]). We shall drop $\det^{\frac{1}{2}}$ -factor in the formulae to simplify notations.

Γ -matrices are the matrix coefficients of a nonzero $\text{Spin}(11)$ -intertwiner $\text{Sym}^2 \mathfrak{s}_{11} \rightarrow V^{11}$, which we call a Γ -map. The components of the Γ -map are C -adjoint to the multiplication map $P' \otimes \Lambda^i P' \rightarrow \Lambda^{i+1} P'$, to the contraction map $P \otimes \Lambda^{i+1} P' \rightarrow \Lambda^i P'$, and to the map $u|_{\Lambda^i P'} = (-1)^i \text{id}$. The action of $\mathfrak{so}_{11} \cong \Lambda^2 V^{11}$ on \mathfrak{s}_{11} is defined in terms of Γ -maps. Let

$$\{f_i\} \subset V^{11} \subset Cl(V^{11}) \tag{46}$$

be an orthonormal basis in V^{11} , $\{\eta_\alpha\}$ be a basis in \mathfrak{s}_{11} which is compatible with the decomposition (43). Then $f_i \times \eta_\alpha \stackrel{\text{def}}{=} \Gamma_{\alpha\gamma}^i C^{\gamma\beta} \eta_\beta$ and

$$f_i \wedge f_j \times \eta_\alpha = \frac{1}{2} (f_i \times (f_j \times \eta_\alpha) - f_j \times (f_i \times \eta_\alpha)) \stackrel{\text{def}}{=} \Gamma_{\alpha\gamma ij} C^{\gamma\beta} \eta_\beta$$

Let e, e' be a basis in U . The element $e \wedge e' \in \mathfrak{sl}_2 \subset \mathfrak{so}_{11}$ is nilpotent. By the elementary representation theory of \mathfrak{sl}_2 the operator in \mathfrak{so}_{11} corresponding to $e \wedge e' \in \mathfrak{so}_{11}$ defines an isomorphism between s^1 and s^{-1} . The matrix of this operator is $\Gamma_{\alpha\gamma ij} C^{\gamma\beta} a^{ij}(U)$. This verifies (26).

Let $e_1, e_2, e_3, e_4, e_5 \in P$ be linearly independent isotropic vectors in V^{11} such that e_1, e_2 span a two dimensional $W \in \text{OGr}(2, 11)$. Then $e_1 \wedge e_2 = a^{ij} f_i \wedge f_j$ and $a^{ij}(W) \Gamma_{\beta ij}^\alpha$ is a matrix of the multiplication operator on $e_1 \wedge e_2$ in $\Lambda[e_1, \dots, e_5] \cong \mathfrak{so}_{11}$. An explicit description of the fiber of $\mathfrak{so}_{\text{OGr}(2, 11)}$ (25) over W is

$$\mathfrak{so}_W \cong e_1 \wedge e_2 \Lambda[e_3, e_4, e_5]. \quad (47)$$

B The Plücker embedding of $\text{OGr}(2, 11)$

In this section we derive equations that characterize the image of the classical Plücker embedding of $\text{OGr}(2, 11)$ into $\mathbf{P}(\Lambda^2 V^{11})$. Let $e_1 \wedge e_2 \in \Lambda^2 V^{11}$ be such that e_1, e_2 span an isotropic space W in V^{11} . The following equations reflects decomposability and isotropy properties of $e_1 \wedge e_2$:

$$\begin{aligned} (e_1 \wedge e_2, e_1 \wedge e_2) &= \frac{1}{2} ((e_1, e_1)(e_2, e_2) - (e_1, e_2)(e_1, e_2)) = 0 \\ e_1 \wedge e_2 \wedge e_1 \wedge e_2 &= 0 \\ \frac{1}{4} ((e_1, e_1)e_2 \circ e_2 + (e_2, e_2)e_1 \circ e_1 - 2(e_1, e_2)e_1 \circ e_2) &= 0 \end{aligned}$$

The symmetric product $e_i \circ e_j$ is an element in $\text{Sym}^2 V^{11}$. We expand e, e' in the orthonormal basis: $e = \sum_{i=1}^{11} a_1^i f_i$, $e' = \sum_{i=1}^{11} a_2^i f_i$. The skew-symmetric matrix $a_{e_1, e_2}^{ij} = a_1^i a_2^j - a_2^j a_1^i$

$$e_1 \wedge e_2 = \sum_{i, j=1}^{11} a_{e_1, e_2}^{ij} f_i \wedge f_j \quad (48)$$

is a function of the basis.

$$a_{Be_1, Be_2}^{ij} = \det(B) a_{e_1, e_2}^{ij}, B \in \text{GL}(2, \mathbb{C}). \quad (49)$$

We see that the coefficients $a^{ij} = a_{e_1, e_2}^{ij}$ have $\text{GL}(2, \mathbb{C})$ -scaling degree one. In other words $a^{ij}(W) = a_{e_1, e_2}^{ij}$ are projective Plücker coordinates of the point

$W = \text{span}(e_1, e_2) \in \text{OGr}(2, 11)$. The matrix a^{ij} satisfies

$$\sum_{i,j=1}^{11} a^{ij} a^{ij} = 0 \quad a^{[ij} a^{kl]} = 0 \quad \sum_{k=1}^{11} a^{ki} a^{kj} = 0 \quad (50)$$

We verify in [27] that 50 are defining equation for $\text{OGr}(2, 11)$ in $\mathbf{P}(\Lambda^2 V^{11})$.

C $\text{Spin}(10, 1, \mathbb{R})$ orbits in $\text{OGr}(2, 11)$

The complex group $\text{SO}(11)$ and its compact form $\text{SO}(11, \mathbb{R})$ act transitively on $\text{OGr}(2, 11)$. The action of the Lorentz group $\text{SO}(10, 1, \mathbb{R})$ has two orbits, which we identify presently.

An isotropic two-dimensional space $W \subset V^{11} = V^{10,1} \otimes \mathbb{C}$ defines a real space $E(W) = (W + \overline{W}) \cap V^{10,1} \otimes \mathbb{R}$. The two numerical invariants of $E(W)$ are dimension and signature $(d(W), \tau(W))$. Invariants $(d(W), \tau(W))$ completely characterize an orbit $O_{d(W), \tau(W)}$. There are two orbits

$$\text{OGr}(2, 11) = \bigcup O_{d(W), \tau(W)} = O_{4,4} \cup O_{3,2}$$

We leave the proof that $O_{4,2}$ is empty to the reader. Here is the idea. Let $W \in O_{4,2}$ then $E(W) \cong \mathbb{R}^2 \times \mathbb{R}^{1,1}$, $W = W \cap \mathbb{R}^2 \otimes \mathbb{C} \times W \cap \mathbb{R}^{1,1} \otimes \mathbb{C}$, but $W \cap \mathbb{R}^{1,1} \otimes \mathbb{C}$ is a complexification of a real isotropic subspace in $\mathbb{R}^{1,1}$. Hence $\dim E(W) = 3$.

The stabilizers of base points in $O_{4,4}$ and $O_{3,2}$ have Lie algebras $\mathfrak{u}_2 \times \mathfrak{so}_{6,1}$ and $\mathfrak{so}_2 \times \mathbb{R} \times \mathfrak{so}_7 \ltimes \mathbb{R}^9$. The orbit $O_{4,4}$ is dense in $\text{OGr}(2, 11)$, the orbit $O_{3,2}$ has the real codimension seven in $\text{OGr}(2, 11)$.

D A Lie algebra description of the flat solution

A CR structure on a homogeneous space has a simple description in terms of the Lie algebra data. We use this idea to characterize the odd twistor transform \mathcal{P} of the flat solution of SUGRA.

We start with a reminder of how to describe a G -invariant CR structure on a homogeneous space G/L in the Lie algebra terms (see [2] for more details). We assume that G is connected. Left-invariant complex vector fields that belong

to the Lie subalgebra $\mathfrak{p} \subset \mathfrak{g}^{\mathbb{C}} = \text{Lie}(G)^{\mathbb{C}}$ define an involutive subbundle $H^{1,0}$ in $T^{\mathbb{C}}(G)$. Let us assume that

$$g\mathfrak{p}g^{-1} \subset \mathfrak{p}, g \in St \quad (51)$$

where St is the stabilizer group of a base point. The subbundle $H^{1,0}$ is invariant with respect to the right St -translations and we can push $H^{1,0}$ to G/St . By construction $H_{G/St}^{1,0}$ is involutive. Obviously, all G -invariant involutive distribution in $T^{\mathbb{C}}(G/St)$ can be obtained this way. The CR structure is nondegenerate if $\mathfrak{p} \cap \bar{\mathfrak{p}} \subset \mathfrak{st}^{\mathbb{C}}$. This construction has a straightforward generalization to supergroups.

In the flat case the distribution F on $\mathbb{R}^{11|32}$ is spanned by (5), and \mathcal{P} is equal to (10).

The super-group of symmetries of the SUGRA datum (\mathcal{M}, F) acts by CR transformations of $\mathcal{P}_{\mathcal{M}, F}$.

The Lie algebra of SP is

$$\mathfrak{so}_{10,1} \ltimes \mathfrak{sush} \quad (52)$$

where \mathfrak{sush} is the algebra of supersymmetries. As a linear space it is a direct sum of $V^{10,1\mathbb{R}}$ (the even part) and $\mathfrak{so}_{10,1\mathbb{R}}$ (the odd part). The only nontrivial bracket is defined by the formula $[\theta, \theta'] = \Gamma(\theta, \theta'), \theta, \theta' \in \mathfrak{so}_{10,1}$. The space $\mathbb{R}^{11|32}$ is the group super-scheme corresponding to Lie algebra \mathfrak{sush} . Vector fields (5) is a basis in the space of odd left-invariant vector fields on $\mathbb{R}^{11|32}$ (see e.g. [15]). This explains why F is invariant under left \mathfrak{sush} translations and infinitesimal rotations by $\mathfrak{so}(10, 1, \mathbb{R})$. These symmetries generate the Lie algebra of SP .

We conclude that the super-Poincaré group SP acts on \mathcal{P} . Vector fields (13) for any given $a^{ij}(W)$ span an abelian Lie subalgebra in \mathfrak{sush} (cf. formulae 16, 17).

The number of orbits of SP in \mathcal{P} coincides the number of orbits $SO(10, 1, \mathbb{R})$ in $\text{OGr}(2, 11)$. The Lie algebra of the stabilizer St of the dense orbit O is isomorphic to $\mathfrak{u}_2 \times \mathfrak{so}_{6,1}$.

The complex Lie algebra \mathfrak{p} , which describes the CR structure on O , is iso-

morphic to the semidirect product $\mathfrak{p}_2 \ltimes \Pi \mathfrak{t}$ where

$$\mathfrak{p}_2 = \text{Ad}(\mathfrak{so}_{11})_0 + \text{Ad}(\mathfrak{so}_{11})_{-1} + \text{Ad}(\mathfrak{so}_{11})_{-2} \text{ see (41) for the notations} \quad (53)$$

and $\mathfrak{t} = s^{-1}$ is as in (43). It coincides with the span of (13) when $a^{ij}(W) = a^{ij}(U)$.

The linear space U is the same as in Appendix A.

The space $\mathcal{P}_{\mathbb{R}^{11|32} \mathbb{C}}$ is a homogeneous space of the complexified super-Poincaré group SP . The isotropy subalgebra $\mathfrak{p} \subset \mathfrak{so}_{11} \ltimes \mathfrak{su} \mathfrak{s} \mathfrak{u} \mathfrak{h} = \text{Lie}(SP)$ of a base point $x \in \mathcal{P}_{\mathbb{R}^{11|32} \mathbb{C}}$ is $\mathfrak{p}_2 \ltimes \Pi \mathfrak{t}$.

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